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Brauer tree algebras and derived equivalence

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Abstract

In this article we introduce the class of the generalised Brauer tree algebras, which is a subclass of the Brauer graph algebras. Our goal is to identify the derived equivalence classes of these algebras. This generalises some previous work by Rickard [9] for blocks with cyclic defect group of a finite modular group algebra. Among the Brauer graph algebras, it is known (see [1] or [2]) that the ones of finite representation type are the Brauer tree algebras. The blocks of cyclic defect of a finite group are Brauer tree algebras.

1. Preliminaries

Throughout this paper every field is assumed to be algebraically closed and is usually denoted by the letter K; we also assume that every K-algebra Λ is finite-dimensional over K. Every Λ-module is a finite-dimensional right module unless stated otherwise. We denote by Mod Λ the category of all Λ-modules and by mod Λ the category of finite-dimensional right Λ-modules. P_Λ is the full subcategory of mod Λ with objects the projective Λ-modules.

We shall freely make use of results on triangulated and derived categories from [7, 12]. In particular, we will use the notation of [7] and we refer to these sources for all proofs of basic facts in this subject.

The following theorem will be specially useful in this paper.

Theorem 1.1 (Rickard [10]). *Let Λ and Γ be two K-algebras. The following conditions are equivalent*:

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1. $K^b(P_\Lambda)$ *and* $K^b(P_\Gamma)$ *are equivalent as triangulated categories.*
2. Γ *is isomorphic to* $\operatorname{End}(T^\bullet)$, *where* $T^\bullet$ *is an object of* $K^b(P_\Lambda)$ *satisfying*
   (a) $\operatorname{Hom}_{D^b(\operatorname{Mod}\Lambda)}(T^\bullet, T^\bullet[i]) = 0$ *for* $i \neq 0$,
   (b) $\operatorname{add}(T^\bullet)$, *the category of direct summands of finite direct sums of copies of* $T^\bullet$, *generates* $K^b(P_\Lambda)$ *as a triangulated category.*

If the equivalent conditions of the theorem are satisfied then we say that Λ and Γ are *derived equivalent* and that $T^\bullet$ is a *tilting complex* for Λ.

2. Brauer graph algebras

In this section we describe some finite-dimensional algebras which arise in the representation theory of finite groups, namely the class of Brauer graph algebras. We first introduce the Brauer graphs.

Definition 2.1 (Benson [2]). A *Brauer graph* consists of a finite undirected connected graph (possibly with loops and multiple edges), together with the following data. To each vertex we assign a cyclic ordering of the edges incident to it, and an integer greater than or equal to one, called the *multiplicity* of the vertex.

A generalised Brauer tree *is a Brauer graph which is a tree.*

A Brauer tree *is a generalised Brauer tree having at most one vertex with multiplicity greater than one.*

Note that at least in the case of a tree, the cyclic ordering on the edges around a vertex is usually indicated by drawing the tree in such a way that the ordering is anti-clockwise around each vertex. Thus, the cyclic orderings are sometimes thought of as being given by a "planar embedding" (see [1] or [2]).

We say that a finite-dimensional K-algebra Λ is a Brauer graph algebra for a given Brauer graph, if there is a one-to-one correspondence between the edges j of the graph and the simple Λ-modules S_j in such a way that the projective cover P_j of S_j has the following description. We have $P_j/\operatorname{Rad}(P_j) \cong \operatorname{Soc}(P_j) \cong S_j$, and $\operatorname{Rad}(P_j)/\operatorname{Soc}(P_j)$ is a direct sum of two (possibly zero) uniserial modules U_j and V_j corresponding to the vertices $u = u_j$ and $v = v_j$ at the ends of the edge j. If the edges around u are cyclically ordered $j, j_1, j_2, \ldots, j_r, j$ and the multiplicity of the vertex u is e_u, then the corresponding uniserial module U_j has composition factors (from the top)

$$S_{j_1}, S_{j_2}, \ldots, S_{j_r}, S_j, S_{j_1}, \ldots, S_{j_r}, S_j, \ldots, \ldots, S_{j_r},$$

so that $S_{j_1}, \ldots, S_{j_r}$ appear e_u times and S_j appears $e_u - 1$ times.

3. Generalised Brauer tree algebras

Let T be a finite (and connected) tree given by a fixed planar representation with n edges and, hence, $n+1$ vertices v_i. We associate with each v_i a positive integer m_i, called the multiplicity of the vertex v_i. We arrange these multiplicities in an $(n+1)$-vector $\bar{m}$, where the entry i is m_i. Then we denote the generalised Brauer tree associated with this data as $(T, \bar{m})$.

It is easy to observe that a generalised Brauer tree $(T, \bar{m})$ determines up to Morita equivalence (isomorphism) a basic K-algebra Λ (see [2]), which is usually symmetric (but always weakly symmetric), called the generalised Brauer tree algebra associated with $(T, \bar{m})$. We sometimes denote the algebra Λ simply by $(T, \bar{m})$. We say that an algebra A is a generalised Brauer tree algebra if its basic algebra Λ is isomorphic to $(T, \bar{m})$ for some generalised Brauer tree T with multiplicity vector $\bar{m}$.

Let A be a generalised Brauer tree algebra and let Λ be its basic algebra and assume Λ is connected. Let Q be the Gabriel quiver of the algebra Λ, then $\Lambda \cong KQ/I$ for some admissible ideal I (see [5]). Given an arrow α in Q we denote by $s(\alpha)$ and $e(\alpha)$ the starting and the ending vertex of α, respectively. By Section 2 we know that A is biserial. Moreover, it is *special biserial* in the following sense.

Definition 3.1. The algebra A is *special biserial* provided its basic algebra $\Lambda = KQ/I$ satisfies the following conditions:

(1) Any vertex of Q is starting point of at most two arrows.
(1*) Any vertex of Q is end point of at most two arrows.
(2) Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\gamma\beta \notin I$.
(2*) Given an arrow γ, there is at most one arrow β with $s(\beta) = e(\gamma)$ and $\gamma\beta \notin I$.

This definition was introduced in [11]. The classification of all the isomorphism classes of indecomposable modules for a special biserial algebra is known (see [3] or [4] for details). Therefore, any special biserial algebra is either *tame* or of *finite type*.

4. A tilting complex

In this section we will generalise the construction of a "canonical" tilting complex by Rickard [9] for Brauer tree algebras (of finite type), to all generalised Brauer tree algebras. We will need the following trivial but important remark on generalised Brauer tree algebras.

Remark 4.1. Let Λ be the K-algebra associated with a generalised Brauer tree $(T, \bar{m})$ and let P_i be the projective Λ-module associated with the edge i. Then

$$\mathrm{Hom}_\Lambda(P_i, P_j) = 0$$

unless the edges i and j have a vertex in common. If i and j have the vertex v in common and $i \neq j$, then $\mathrm{Hom}_\Lambda(P_i, P_j)$ is e_v-dimensional.

If the edge i has vertices v and u then $End(P_i)$ is $e_v + e_u$-dimensional.

Given a generalised Brauer tree $(T, \bar{m})$ with n edges and, therefore, $n+1$ vertices, we choose (or distinguish) an arbitrary but fixed vertex v of T. We define v to be the *root* of the tree. Let i be an edge of T, we denote by u_i and v_i the closest and the furthest end of i from the root v, respectively.

Let Λ be the generalised Brauer graph algebra associated with $(T, \bar{m})$. For each edge i of T there is a unique path in T from the root v to the furthest end of i. This defines a sequence

$$i_0, i_1, \ldots, i_r = i$$

of edges. By the above Remark 4.1, one has $\mathrm{Hom}(P_{i_s}, P_{i_{s+1}})$ is e_{v_s}-dimensional, where v_s is the vertex in common between i_s and i_{s+1}, for all $s = 0, \ldots, r-1$. For every $s = 0, \ldots, r-1$ choose an homomorphism

$$\psi_s : P_{i_s} \longrightarrow P_{i_{s+1}}$$

with minimal kernel (or maximal image). That is, ψ_s is a projective cover of the unique uniserial submodule of $P_{i_{s+1}}$ with top factor S_{i_s} and e_{v_s} composition factors isomorphic to S_{i_s}. Hence, ψ_s is defined up to automorphism. We may describe ψ_s more precisely. At the vertex v_s of T consider the succession of edges $i_{s+1} = j_0, j_1, \ldots, j_t = i_s$ in the cyclic ordering at v_s from i_{s+1} to i_s without repetition of edges. Let x_{j_i} be the set of corresponding vertices in the quiver of the algebra Λ. Then there exists a succession of arrows $\gamma_i : x_{j_i} \longrightarrow x_{j_{i+1}}$, $0 \le i \le t-1$. The path defined by these arrows corresponds to a non-zero monomial $\Gamma = \gamma_0\gamma_1 \cdots \gamma_{t-1}$ in the algebra Λ. Then the map $\psi_s : P_{i_s} \longrightarrow P_{i_{s+1}}$ may be given by multiplication on the left by Γ.

Therefore, there is a unique, up to isomorphism (homotopy), complex of projective Λ-modules

$$\cdots \longrightarrow 0 \longrightarrow P_{i_0} \longrightarrow P_{i_1} \longrightarrow \cdots \longrightarrow P_{i_r} \longrightarrow 0 \longrightarrow \cdots,$$

where all the maps are of the form ψ_s (hence, non-zero), $s = 0, \ldots, r-1$, and where P_{i_0} is the degree zero term. This is easy to prove by induction on r and using a mapping cone argument. Let $C(i)$ denote this complex considered as an object of $D^{\mathrm{b}}(P_\Lambda)$.

Let C be the direct sum of the complexes $C(i)$'s, where i represents an edge of T. Note that the number of summands of C is the same as the number of simple Λ-modules. Our next step is to show that C is indeed a tilting complex for Λ.

Lemma 4.2. *The object C in $D^{\mathrm{b}}(P_\Lambda)$ is a tilting complex for Λ.*

Proof. We first show that the category add(C), the full subcategory of $D^{\mathrm{b}}(P_\Lambda)$ consisting of direct summands of direct sums of copies of C, generates $D^{\mathrm{b}}(P_\Lambda)$ as a triangulated category. It is enough to show that the projectives Λ-modules P_i (one for each edge i of T) are in add(C) as stalk complexes in degree zero (see [7]). Denote this complex as $P_i^\bullet$. Let $i_0, i_1, \ldots, i_r = i$ be the unique sequence of edges from the root v to the furthest end v_i of i. If $i = i_0$ then $P_i^\bullet$ is in add(C) and we have nothing to prove. If $r > 0$, let $\psi_{r-1}^\bullet : C(i_{r-1})[-1] \to P_i^\bullet[-r]$ be given by zero maps in degree different from r, and in this degree it is given by $\psi_{r-1} : P_{i_{r-1}} \to P_{i_r}$. This map can be embedded into a triangle whose third term is isomorphic to the mapping cone of the map $\psi^{\bullet}_{r-1}$, which is clearly isomorphic to $C(i_r) = C(i)$. Thus, we obtain a triangle with two terms in add(C), hence the third, namely $P_i^{\bullet}[-r]$, is in the triangulated category generated by add(C), for all edge i. Using the translation functor, it is clear that $P_i^{\bullet}$ is in add(C) for all i.

It remains to show that for $t \neq 0$, $\mathrm{Hom}(C, C[t]) = 0$ (see Theorem 1.1). By the Remark 4.1 above, we have that $\mathrm{Hom}(C, C[t]) = 0$ unless t is -1, 0 or 1.

Consider a map α of complexes from $C(i)$ to $C(j)[1]$. This consists of maps

$$\alpha_m : P_{i_m} \longrightarrow P_{j_{m+1}}$$

making the following diagram commute:

$$\begin{array}{ccccccccc}
 & & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow & \cdots \\
 & & {\scriptstyle \alpha_0}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\
P_{j_0} & \longrightarrow & P_{j_1} & \longrightarrow & P_{j_2} & \longrightarrow & P_{j_3} & \longrightarrow & \cdots .
\end{array}$$

If $\alpha \neq 0$, then we can choose s as large as possible so that $\alpha_s \neq 0$. We may assume that we have chosen α from its homotopy class so as to minimise this value of s. By the above Remark 4.1 we have that $i_s = j_s$, since the contrary assumption implies the existence of two different paths from the root v to the furthest end of j_{s+1}. Moreover, arguing by induction, we know that $i_t = j_t$ for all $0 \leq t \leq s$.

Therefore, α_s factors through $\psi_s : P_{j_s} \to P_{j_{s+1}}$, since the image of α_s is contained in the image of ψ_s up to homotopy, by the choice of ψ_s. But this factoring map $P_{i_s} \to P_{j_s}$ gives a homotopy from α to a map β for which $\beta_t = 0$ for $t \geq s$. Thus, α must be zero and therefore

$$\mathrm{Hom}(C, C[1]) = 0.$$

Now consider a map α of complexes from $C(i)$ to $C(j)[-1]$. Again this consists of maps

$$\alpha_m : P_{i_{m+1}} \longrightarrow P_{j_m}$$

making the following diagram commute:

$$\begin{array}{ccccccccc}
P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow & P_{i_3} & \longrightarrow & \cdots \\
\downarrow & & {\scriptstyle \alpha_0}\downarrow & & {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & \\
0 & \longrightarrow & P_{j_0} & \longrightarrow & P_{j_1} & \longrightarrow & P_{j_2} & \longrightarrow & \cdots .
\end{array}$$

If $\alpha \neq 0$, then we can choose s as small as possible so that $\alpha_s \neq 0$. By the above Remark 4.1 we have that $i_s = j_s$, since the contrary assumption implies the existence of two different paths from the root v to the furthest end of i_{s+1}. Hence, by induction and using that T is a tree, we know that $i_t = j_t$ for all $0 \leq t \leq s$.

Note that the composition

$$P_{i_s} \xrightarrow{\psi_s} P_{i_{s+1}} \xrightarrow{\alpha_s} P_{j_s}$$

has image containing the socle of P_{j_s} if $\alpha_s \neq 0$ up to homotopy, by the choice of ψ_s. In particular, this composition is non-zero, contradicting the fact that the diagram commutes. Thus, α must be zero and therefore

$$\mathrm{Hom}(C, C[-1]) = 0.$$

This finishes the proof. □

5. The algebra End(C)

In the rest of this article we will be interested in the algebra of endomorphisms End(C) in $D^{\mathrm{b}}(P_\Lambda)$ of the tilting complex C. Since we write maps on the right, we should take $(\mathrm{End}(C))^{\mathrm{op}}$; however, this would complicate our calculations unnecessarily. So we decided to work in End(C) and warn the reader about our convention.

The algebra End(C) is derived equivalent to the fixed generalised Brauer tree algebra Λ with Brauer tree $(T, \bar{m})$, by Theorem 1.1. In this section we begin the description of End(C) in order to prove the main theorem of this article.

Remember that the Cartan invariants of End(C) are given as follows:

$$c_{ij} = \dim_K \mathrm{Hom}(C(i), C(j)).$$

These are easy to calculate using Remark 4.1 and the fact that for any two objects $P_a^\bullet$ and $P_b^\bullet$ of $K^{\mathrm{b}}(P_\Lambda)$ we have the formula (see [7, III.1.4; 9])

$$\sum_n (-1)^n \dim_K \mathrm{Hom}(P_a^\bullet, P_b^\bullet[n]) = \sum_{r,s} (-1)^{r-s} \dim_K \mathrm{Hom}(P_a^r, P_b^s), \tag{1}$$

and for $C(i)$ and $C(j)$ the left-hand side reduces to c_{ij} (see Lemma 4.2). Given an edge i denote by v_i the furthest end of i from the root v, and by e_i its multiplicity. The multiplicity of the root v is e_v.

Lemma 5.1. *The Cartan invariants of* End(C) *are given as follows*:

$$c_{ij} = \begin{cases} e_v + e_i & \text{if } i = j, \\ e_v & \text{otherwise.} \end{cases}$$

Proof. Let i be the edge corresponding to $C(i)$ and let $i_0, i_1, \ldots, i_k = i$ be the unique sequence of edges defined by the unique path in T from the root v to the furthest end of i. Let $j_0, j_1, \ldots, j_l = j$ be the corresponding sequence for the edge j. Then the right hand-side of Eq. (1) above is zero unless $r = s$, $r = s + 1$ or $r = s - 1$ by Remark 4.1. Assume $i \neq j$ and suppose $i_t = j_t$ for $0 \leq t \leq m$, for some $m \leq k, l$ and $i_t \neq j_t$ otherwise. When $r = s$ one gets $(e_v + e_{i_0}) + (e_{i_0} + e_{i_1}) + \cdots + (e_{i_{m-1}} + e_{i_m}) + e_{i_m}$. When

$r = s + 1$ or $r = s - 1$ we obtain $-e_{i_0} - e_{i_1} - \cdots - e_{i_{m-1}} - e_{i_m}$. Adding the three terms together we have $c_{ij} = e_v$ if $i \neq j$.

Now we consider the case $i = j$, then when $r = s$ we have $(e_v + e_{i_0}) + (e_{i_0} + e_{i_1}) + \cdots + (e_{i_{k-2}} + e_{i_{k-1}}) + (e_{i_{k-1}} + e_{i_k})$. When $r = s + 1$ or $r = s - 1$ we get $-e_{i_0} - e_{i_1} - \cdots - e_{i_{k-2}} - e_{i_{k-1}}$. Adding the three terms together we have $c_{ii} = e_v + e_i$. □

5.1. *The algebra* End($C(i)$)

We first study the endomorphism ring of each direct summand of C. Let

$$C(i): \ \cdots 0 \to \cdots \to P_{i_0} \to P_{i_1} \to \cdots \to P_{i_r} \to 0 \cdots$$

be a direct summand of C, where P_{i_0} is the degree zero term and $i = i_r$. We know that $\dim_K \mathrm{End}(C(i)) = e_v + e_i$, where v is the root of T and v_i is the furthest end of i, by Lemma 5.1.

We will give a basis for End($C(i)$). Remember from Section 2 that the quotient $\mathrm{Rad}(P_{i_0})/\mathrm{Soc}(P_{i_0})$ is a direct sum of two (possibly zero) uniserial modules U_{i_0} and V_{i_0}, corresponding to the vertices v and v_{i_0} at the ends of the edge i_0. The edges around the root v are cyclically ordered $i_0 = j, j_1, j_2, \ldots, j_s, j$ and the multiplicity of the vertex v is e_v. Thus, the corresponding uniserial module U_{i_0} has composition factors (from the top)

$$S_{j_1}, S_{j_2}, \ldots, S_{j_s}, S_j, S_{j_1}, \ldots, S_{j_s}, S_j, \ldots, \ldots, S_{j_s},$$

so that $S_{j_1}, \ldots, S_{j_s}$ appear e_v times and S_j appears $e_v - 1$ times.

Let $\zeta \in \mathrm{End}(P_{i_0})$ be the function corresponding to the first composition factor (from the top) of U_{i_0} isomorphic to $S_{i_0} = S_j$. That is, ζ is a projective cover of the uniserial submodule of P_{i_0}, having composition factors (from the top)

$$S_j, S_{j_1}, S_{j_2}, \ldots, S_{j_s}, S_j, S_{j_1}, \ldots, S_{j_s}, S_j, \ldots, \ldots, S_{j_s}, S_j,$$

so that $S_j, S_{j_1}, \ldots, S_{j_s}$ appear $e_v - 1$ times and S_j appears e_v times.

If U_{i_0} is zero then take ζ the projection of P_{i_0} onto its socle. Note that ζ is the projection of P_{i_0} onto its socle when $e_v = 1$ (and U_{i_0} may be not zero). With the function ζ we construct a function $\zeta^{\bullet} \in \mathrm{End}(C(i))$ given as follows:

$$\begin{array}{ccccccccccccccc}
C(i): & \cdots & \longrightarrow & 0 & \longrightarrow & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow \cdots \longrightarrow & P_{i_r} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \zeta^{\bullet}} & & & \downarrow & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
C(i): & \cdots & \longrightarrow & 0 & \longrightarrow & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow \cdots \longrightarrow & P_{i_r} & \longrightarrow & \cdots
\end{array}$$

where the function $\zeta^{\bullet}$ is zero in degrees different from 0, and in this degree it is given by ζ. It is clear that $\zeta^{\bullet}$ is a map of complexes by Remark 4.1. Note that U_{i_0} is on the "side" of P_{i_0} corresponding to the root v. Let $\zeta_k^{\bullet} = (\zeta^{\bullet})^k$ for $k = 1, \ldots, e_v$. Then, for instance, $\zeta^{\bullet}_{e_v}$ is the projection onto the socle of the first component P_{i_0} and zero in the other degrees. It is clear that these functions $\zeta^{\bullet}_k$, $k = 1,\dots,e_v$ form a linear independent subset of $\operatorname{End}(C(i))$.

Now we look at the other end of our complex $C(i)$, namely the projective P_{i_r}, where $i_r = i$. Remember from Section 2 that $\operatorname{Rad}(P_{i_r})/\operatorname{Soc}(P_{i_r})$ is a direct sum of two (possibly zero) uniserial modules U_{i_r} and V_{i_r}, corresponding to the closer end u_{i_r} and the furthest end v_{i_r} of the edge i_r from the root v, respectively. The edges around v_{i_r} are cyclically ordered $i_r = j, j_1, j_2, \dots, j_t, j$ and the multiplicity of the vertex v_{i_r} is e_i. Thus, the corresponding uniserial module V_{i_r} has composition factors (from the top)

$$S_{j_1}, S_{j_2}, \dots, S_{j_t}, S_j, S_{j_1}, \dots, S_{j_t}, S_j, \dots, \dots, S_{j_t},$$

so that $S_{j_1}, \dots, S_{j_t}$ appear e_i times and S_j appears $e_i - 1$ times.

Let $\eta \in \operatorname{End}(P_{i_r})$ corresponding to the first composition factor (from the top) of V_{i_r} isomorphic to $S_{i_r} = S_j$. If V_{i_r} is zero then take η the projection of P_{i_r} onto its socle. Note that if $e_i = 1$ then η is the projection of P_{i_r} onto its socle. With the function η we construct a function $\eta^{\bullet} \in \operatorname{End}(C(i))$, given as follows:

$$\begin{array}{ccccccccccccccc}
C(i): & \cdots \to & 0 & \to & P_{i_0} & \to & P_{i_1} & \to & P_{i_2} & \to & \cdots & \to & P_{i_r} & \to & \cdots \\
\downarrow \eta^{\bullet} & & \downarrow & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & & & \downarrow \eta & & \\
C(i): & \cdots \to & 0 & \to & P_{i_0} & \to & P_{i_1} & \to & P_{i_2} & \to & \cdots & \to & P_{i_r} & \to & \cdots
\end{array}$$

where $\eta^{\bullet}$ is given by the function η in degree r and zero otherwise. It is clear that $\eta^{\bullet}$ is a map of complexes by Remark 4.1. Note that V_{i_r} is on the "side" of P_{i_r} corresponding to the furthest end v_{i_r} of i_r from the root v. Let $\eta^{\bullet}_k = (\eta^{\bullet})^k$ for $k = 1,\dots,e_i$. Then, for example, $\eta^{\bullet}_{e_i}$ is the projection onto the socle of the last component P_{i_r} and zero in the other degrees. It is clear that these functions $\eta^{\bullet}_k$, $k = 1,\dots,e_i$, form a linear independent subset of $\operatorname{End}(C(i))$. From the definition of the functions $\zeta^{\bullet}$ and $\eta^{\bullet}$ it is plain that

$$\zeta^{\bullet}\eta^{\bullet} = \eta^{\bullet}\zeta^{\bullet} = 0. \tag{2}$$

For every $1 \le k \le r$ we have a map $h_k : P_{i_k} \to P_{i_{k-1}}$ corresponding to the lowest composition factor isomorphic to S_{i_k} in $P_{i_{k-1}}$ (see Section 3). The map $h^{\bullet} : C(i) \to C(i)$ defined by the maps h_k, gives a chain homotopy between $\zeta^{\bullet}_{e_v}$ and $\eta^{\bullet}_{e_i}$:

$$\begin{array}{ccccccccccccccc}
C(i): \cdots & 0 & \to & P_{i_0} & \to & P_{i_1} & \to & P_{i_2} & \to \cdots \to & P_{i_{r-1}} & \to & P_{i_r} & \cdots \\
\downarrow h^{\bullet} & \downarrow & & \downarrow \zeta^{e_v} & \overset{h_1}{\swarrow} & \downarrow 0 & \overset{h_2}{\swarrow} & \downarrow 0 & & \downarrow 0 & \overset{h_r}{\swarrow} & \downarrow \eta^{e_i} & \\
C(i): \cdots & 0 & \to & P_{i_0} & \to & P_{i_1} & \to & P_{i_2} & \to \cdots \to & P_{i_{r-1}} & \to & P_{i_r} & \cdots
\end{array}$$

In fact, remember from Section 4 that the maps $\psi_s : P_{i_s} \to P_{i_{s+1}}$, that build up the complex $C(i)$, were chosen such that the kernel is minimal. That is, ψ_s corresponds to the highest composition factor of $P_{i_{s+1}}$ isomorphic to S_{i_s} (the top of P_{i_s}). Therefore,

$\psi_s h_{s+1}$ and $h_s\psi_{s-1}$ are both maps from P_{i_s} onto its socle. Then we may chose $h^\bullet$ so that $\psi_s h_{s+1} = h_s\psi_{s-1}$, $s = 1,\dots,r-2$, and $\zeta^{e_v} = \psi_0 h_1$, $\eta^{e_i} = h_r\psi_{r-1}$. It is now shown that in $\mathrm{End}(C(i))$

$$\zeta^\bullet_{e_v} = \eta^\bullet_{e_i}. \tag{3}$$

We summarize the calculations above in the following proposition.

Proposition 5.2. *Let $C(i)$ be a direct summand of C and let $\zeta^\bullet$ and $\eta^\bullet$ as above. Then $\dim_K \mathrm{End}(C(i)) = e_v + e_i$ and a basis of this space is given by the powers of $\zeta^\bullet$ and $\eta^\bullet$ together with the identity map.*

Proof. It is plain that the sets $\{\zeta^\bullet_k : k = 0,\dots,e_v\}$ and $\{\eta^\bullet_k : k = 0,\dots,e_i\}$, where $\zeta^\bullet_0 = \eta^\bullet_0$ is the identity map, are linear independent sets. Note that any map $\varphi^\bullet$ homotopic to $\zeta^\bullet_j$, with $0 \le j < e_v$, has as its degree zero function a scalar multiple of ζ^j. Analogously, any map $\varphi^\bullet$ homotopic to $\eta^\bullet_j$, with $0 \le j < e_i$, has its degree r function a scalar multiple of η^j. Therefore,

$$\{\zeta^\bullet_0, \zeta^\bullet_1, \dots, \zeta^\bullet_{e_v}, \eta^\bullet_1, \dots, \eta^\bullet_{e_i-1}\}$$

is a linear independent set (note that $\zeta^\bullet_{e_v} = \eta^\bullet_{e_i}$), and its cardinality is $e_v + e_i$ which is the dimension of $\mathrm{End}(C(i))$ by Lemma 5.1. □

We can say more about the structure of $\mathrm{End}(C(i))$.

Proposition 5.3. *Let $C(i)$ be a direct summand of C. Then $\mathrm{End}(C(i))$ is a local commutative special biserial algebra.*

Proof. The quiver Q of $\mathrm{End}(C(i))$ has one vertex. If $e_v = e_i = 1$ then Q has no arrows and the proposition is true. If $e_i = 1$ and $e_v > 1$ then Q has only one arrow α corresponding to $\zeta^\bullet$ and the only relation is $\alpha^{e_v+1} = 0$, then the proposition is true. Symmetrically, if $e_i > 1$ and $e_v = 1$ then Q has only one arrow β corresponding to $\eta^\bullet$ and the only relation is $\beta^{e_i+1} = 0$. If $e_i > 1$ and $e_v > 1$ then the quiver of $\mathrm{End}(C(i))$ has two arrows α and β corresponding to $\zeta^\bullet$ and $\eta^\bullet$, respectively. The relations from Eqs. (2) and (3) imply $\alpha\beta = \beta\alpha = 0$ and $\alpha^{e_v} = \beta^{e_i}$. Hence, $\mathrm{End}(C(i))$ is special biserial (see Definition 3.1) and clearly commutative. □

Therefore, $\mathrm{End}(C(i))$ is a generalised Brauer tree algebra, with generalised Brauer tree

$e_v \bullet$ ———————— $\bullet\ e_i$.

Remark 5.4. Let Λ be a Brauer graph algebra with graph G. Let T be a subgraph of G consisting of a subset of vertices of G and all the edges between them. Assume T is a connected tree, then T is in fact a generalised Brauer tree and the complexes $C(i)$

constructed above (with respect to a some fixed root of T) satisfy that its endomorphism ring $\mathrm{End}(C(i))$ is a local commutative special biserial algebra (as in Proposition 5.3).

6. The quiver of End(C)

Let Λ be a generalised Brauer tree algebra with generalised Brauer tree $(T, \bar{m})$. In Section 4 we constructed from this data, a tilting complex called C. In this section we will describe the Gabriel quiver Q_C of the algebra $\mathrm{End}(C)$. In order to know that, we will be interested in the space $\mathrm{Hom}(C(i), C(j))$ for $i \neq j$, and $C(i)$, $C(j)$ direct summands of C. We know that the Gabriel quiver Q_C has n points, each one corresponding to each edge of T. That is, each one corresponding to each indecomposable summand $C(i)$ of C. We want to know the arrows between them, using the basis of $\mathrm{End}(C(i))$ for each i constructed in Section 5 (see Proposition 5.2).

We make an abuse of notation denoting by i the vertex in the quiver Q_C corresponding to the summand $C(i)$ of C and the edge i of T. We want to know the number of arrows between the vertices i and j. Let $\mathrm{R\,Hom}(C(i), C(j))$ be the subspace of $\mathrm{Hom}(C(i), C(j))$ generated by the homomorphisms $\alpha^\bullet : C(i) \to C(j)$ so that $\alpha^\bullet$ is not an isomorphism in $D^b(P_\Lambda)$. Let $\mathrm{R}^2\,\mathrm{Hom}(C(i), C(j))$ be the subspace of $\mathrm{R\,Hom}(C(i), C(j))$ generated by all the homomorphisms $\delta^\bullet : C(i) \to C(j)$ such that $\delta^\bullet = \alpha^\bullet\beta^\bullet$, where $\alpha^\bullet$ is in $\mathrm{R\,Hom}(C(i), C(k))$ and $\beta^\bullet$ is in $\mathrm{R\,Hom}(C(k), C(j))$ for some direct summand $C(k)$ of C. Then the number of arrows from i to j is (see for example [2, 4], or [5])

$$\dim_K \frac{\mathrm{R\,Hom}(C(i), C(j))}{\mathrm{R}^2\,\mathrm{Hom}(C(i), C(j))}.$$

Fix an edge i and let $\eta^\bullet, \zeta^\bullet \in \mathrm{End}(C(i))$ as in the Proposition 5.2.

Lemma 6.1. *If the multiplicity e_i of the vertex v_i is greater than one then*

$$\eta^\bullet \in \mathrm{R\,End}(C(i)) \setminus \mathrm{R}^2\,\mathrm{End}(C(i)).$$

Proof. Remember that the edge $i = i_r$ defines a unique path $i_0, i_1, \ldots, i_r$ from the root v to v_i. Then the complex $C(i)$ is as follows:

$$C(i): \ \cdots 0 \to \cdots \to P_{i_0} \to P_{i_1} \to \cdots \to P_{i_r} \to 0 \cdots.$$

Assume for a contradiction that $\eta^\bullet \in \mathrm{R}^2\,\mathrm{End}(C(i))$. Then $\eta^\bullet = \sum_k \alpha_k^\bullet \beta_k^\bullet$, where the sum runs over all edges of the Brauer tree T, and for each edge k, $\alpha_k^\bullet \in \mathrm{Hom}(C(i), C(k))$ and $\beta_k^\bullet \in \mathrm{Hom}(C(k), C(i))$. Let η be the function in degree r of $\eta^\bullet$, then η is not the projection of P_{i_r} onto its socle by hypothesis. Therefore, $\eta^\bullet$ cannot be homotopic to a map $\gamma^\bullet$ with its term in degree r different from η, up to an automorphism of P_{i_r}. Therefore, the corresponding maps of degree r $\sum_k (\alpha_k^\bullet)_r (\beta_k^\bullet)_r = \eta$ (up to an automorphism). Then, there exists an edge k so that $(\alpha_k^\bullet)_r (\beta_k^\bullet)_r = \eta$, up to an automorphism, and the succession of edges from the root v to the furthest end of k contains the corresponding path defined by the edge i, since the map η cannot be factorised through a projective P_{k_r} with $k_r \neq i_r$. That is, the path from the root v to i is contained in the path from v to k since T is a tree.

If the path from the root v to i is properly contained in the corresponding path to k, let $\psi_r : P_i \to P_{k_{r+1}}$ be the function at level r of the complex $C(k)$ as in Section 4. Then $(\alpha_k^\bullet)_r \psi_r \neq 0$, forming a contradiction since $e_i > 1$. Therefore $C(k) = C(i)$. Considering $\mathrm{End}(P_i)$ as an algebra, we have

$$\eta \in \mathrm{R}\ \mathrm{End}(P_i) \setminus \mathrm{R}^2\ \mathrm{End}(P_i), \tag{4}$$

then we find that $\alpha_r = (\alpha_k^\bullet)_r$ or $\beta_r = (\beta_k^\bullet)_r$ is an automorphism. Suppose α_r is an automorphism. Then α_{r-1} is also an automorphism of $P_{i_{r-1}}$ by the choice of ψ_{r-1} (see Section 4), since the diagram must commute. Arguing by induction we see that α_j is some automorphism for all j. Therefore, $\alpha_k^\bullet$ is an isomorphism and hence $\eta^\bullet \notin \mathrm{R}^2\ \mathrm{End}(C(i))$, a contradiction. If β_r is an automorphism then β_{r-1} is also an automorphism, since $\beta_k^\bullet$ is a map of complexes. Again, by induction, the maps β_j are all automorphisms. Therefore, $\beta_k^\bullet$ is an isomorphism and hence $\eta^\bullet \notin \mathrm{R}^2\ \mathrm{End}(C(i))$, a contradiction. This finishes the proof. □

Corollary 6.2. *Let i be the vertex in Q_C corresponding to the edge i. If the corresponding multiplicity $e_i > 1$ then Q_C has a loop at i.*

From the corollary it is clear that the choice of the root v of the generalised Brauer tree T is determinant for the structure of the algebra $\Lambda = \mathrm{End}(C)$. We observe that loops do not occur in Rickard's paper [9], since he takes as the root of a Brauer tree the unique exceptional vertex of T. In our more general case, the quiver of Λ has loops whenever there exists a vertex with multiplicity greater than one and it is different from the root of the tree.

We have seen that the function $\eta^\bullet$ (see Proposition 5.2) determines some arrows in the quiver Q_C. We will use the other function $\zeta^\bullet$ as in Proposition 6.1 to find out the remaining arrows.

6.1. A cyclic ordering on T

In this section we find a cyclic ordering on the complexes $C(i)$, which correspond to a cyclic ordering on the edges of T. Recall that the complex $C(i)$ corresponds to a unique path or succession of edges in the generalised Brauer tree T as follows:

$$\underset{v}{\bullet} \stackrel{i_0}{\text{——}} \underset{v_0}{\bullet} \stackrel{i_1}{\text{——}} \underset{v_1}{\bullet} \cdots \underset{v_{r-2}}{\bullet} \stackrel{i_{r-1}}{\text{——}} \underset{v_{r-1}}{\bullet} \stackrel{i_r}{\text{——}} \underset{v_r}{\bullet}.$$

We define the successor of $C(i)$ as the complex corresponding to the succession

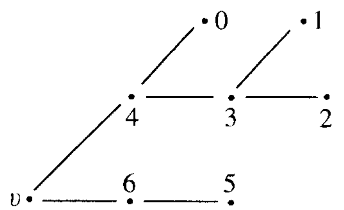
$$\underset{v}{\bullet} \stackrel{i_0}{\text{——}} \underset{v_0}{\bullet} \stackrel{i_1}{\text{——}} \underset{v_1}{\bullet} \cdots \underset{v_{r-2}}{\bullet} \stackrel{i_{r-1}}{\text{——}} \underset{v_{r-1}}{\bullet} \stackrel{j_r}{\text{——}} \underset{v'_r}{\bullet} \stackrel{j_{r+1}}{\text{——}} \underset{v'_{r+1}}{\bullet} \cdots \underset{v'_{t-1}}{\bullet} \stackrel{j_t}{\text{——}} \underset{v'_t}{\bullet} \cdots,$$

where the edge j_r is the predecessor of i_r in the cyclic ordering at the vertex v_{r-1} and $j_r \neq i_{r-1}$. The edge j_{r+s} is the predecessor of $j_{r+(s-1)}$ at $v'_{r+(s-1)}$ for all $s \geq 1$ up to j_t which is a "terminal" edge. That is, j_t is the only edge adjacent to v'_t.

If the predecessor of i_r is i_{r-1} then we define the successor of $C(i) = C(i_r)$ as $C(i_{r-1})$. Note that if T is a *star* then the cyclic ordering just defined is clockwise around the root v.

Lemma 6.3. *The above definition of successor for every complex $C(i)$ gives rise to a cyclic ordering on the set of complexes $C(i)$'s.*

Let us consider an example. Let T be the following Brauer tree:



where the cyclic ordering on the edges around a vertex is anticlockwise, indicated by the drawing of the tree and the vertex v is the root. The numbers attached to each vertex different from the root shows the cyclic order on the complexes associated with the corresponding succession of edges from the root. This cyclic order only depends on the shape and on the cyclic order in each vertex of the Brauer tree and it is independent of their multiplicities.

6.2. *The remaining arrows of Q_C*

We define a map of complexes from the complex $C(i) = C(i_r)$ (where the edge i is arbitrary but fixed as above) to its successor in the cyclic ordering on the complexes (or edges) that we have just defined. If the successor of $C(i_r)$ is $C(i_{r-1})$, we define the map of complexes $\alpha^\bullet : C(i_r) \to C(i_{r-1})$ as the identity in degrees $0, \dots, r-1$ and zero otherwise. This is clearly a map of complexes.

If the successor of $C(i_r)$ is $C(j_t)$, where the complex $C(j_t)$ is as above, we define the map of complexes $\alpha^\bullet : C(i_r) \to C(j_t)$ as the identity in degrees $0, \dots, r-1$. In degree r we know that the top of the radical of P_{j_r} has a summand isomorphic to S_{i_r} the top of P_{i_r}, since j_r is the edge which precedes i_r at the vertex v_{r-1}, the furthest end of i_{r-1}. We define α_r as the function corresponding to this composition factor of P_{j_r}; clearly this choice (up to an automorphism) makes $\alpha^\bullet$ a map of complexes. In other words, the function α_r is chosen in such a way to force $\alpha^\bullet$ to be a map of complexes, given that α_i is the identity for all $0 \leq i \leq r-1$. The functions of the form $\alpha^\bullet$ will give us the remaining arrows in the quiver Q_C of $\mathrm{End}(C)$.

Lemma 6.4. *Let $C(k)$ be the successor of $C(i_r)$ (k is either i_{r-1} or j_t, depending on the cyclic ordering) and define $\alpha^\bullet : C(i_r) \to C(k)$ as above. Then*

$$\alpha^\bullet \in \mathrm{R}\ \mathrm{Hom}(C(i_r), C(k)) \setminus \mathrm{R}^2\ \mathrm{Hom}(C(i_r), C(k)).$$

Proof. Assume for a contradiction that $\alpha^\bullet \in \mathrm{R}^2\ \mathrm{Hom}(C(i_r), C(k))$. That is, there exists a factorization of $\alpha^\bullet$ of the form $\alpha^\bullet = \sum_l \beta_l^\bullet \gamma_l^\bullet$, where the sum runs over all edges of the generalised Brauer tree, and for each edge l, the functions $\beta_l^\bullet \in \mathrm{R}\ \mathrm{Hom}(C(i_r), C(l))$ and $\gamma_l^\bullet \in \mathrm{R}\ \mathrm{Hom}(C(l), C(k))$. Since the functions $\alpha_0, \alpha_1, \dots, \alpha_{r-1}$ are identity maps, any map $\psi^\bullet$ homotopic to $\alpha^\bullet$ satisfies that $\psi_j^\bullet$ is an automorphism for all $0 \le j \le r-1$. Therefore, $\sum_l (\beta_l^\bullet)_j (\gamma_l^\bullet)_j$ is an automorphism for all $0 \le j \le r-1$. In particular, there exists an edge l such that $(\beta_l^\bullet)_{r-1}$ and $(\gamma_l^\bullet)_{r-1}$ are both automorphisms. Therefore, $C(l)$ has a subcomplex isomorphic to $C(i_{r-1})$.

Let first consider the case where the successor of $C(i_r)$ is $C(i_{r-1})$. If $C(l)$ is isomorphic to $C(i_{r-1})$ then $\gamma_l^\bullet$ is an isomorphism, giving a contradiction. So we may assume that $C(l)$ properly contains a subcomplex isomorphic to $C(i_{r-1})$. Let P_{l_r} be the component in degree r of the complex $C(l)$, if $P_{l_r} \ncong P_{i_r}$ then the function

$$\varphi'_{r-1} : P_{l_{r-1}} \cong P_{i_{r-1}} \to P_{l_r}$$

in degree $r-1$ of the complex $C(l)$ would factor as $\varphi_{r-1}(\beta_l^\bullet)_r$, which implies that φ'_{r-1} maps the top of $P_{l_{r-1}}$ into the second composition factor isomorphic to $S_{l_{r-1}}$ from the top of P_{l_r}, since by hypothesis $C(i_{r-1})$ is the successor of $C(i_r)$, a contradiction. Therefore, $P_{l_r} \cong P_{i_r}$ and the map $(\beta_l^\bullet)_r$ is an isomorphism, by the same argument. Since $\beta_l^\bullet$ is a map of complexes, we find that $C(i_r) \cong C(l)$ and $\beta_l^\bullet$ is an isomorphism of complexes, this contradicts the assumption that $\alpha^\bullet \in \mathrm{R}^2\ \mathrm{Hom}(C(i_r), C(k))$. Therefore,

$$\alpha^\bullet \in \mathrm{R}\ \mathrm{Hom}(C(i_r), C(i_{r-1})) \setminus \mathrm{R}^2\ \mathrm{Hom}(C(i_r), C(i_{r-1}))$$

when k is i_{r-1}.

We consider the remaining case, namely the case where the successor of $C(i_r)$ is $C(j_t)$. Since the edge j_r is the predecessor of i_r in the cyclic ordering of the vertex v_{r-1} (the furthest end of the edge i_{r-1}), the map

$$\alpha_r \in \mathrm{R}\ \mathrm{Hom}(P_{i_r}, P_{j_r}) \setminus \mathrm{R}^2\ \mathrm{Hom}(P_{i_r}, P_{j_r}).$$

Then $\sum_l (\beta_l^\bullet)_r (\gamma_l^\bullet)_r$ is $\alpha_r^\bullet$ up to an automorphism. Therefore, for some edge l, $\beta_r = (\beta_l^\bullet)_r$ or $\gamma_r = (\gamma_l^\bullet)_r$ is an isomorphism. If β_r is an isomorphism then $P_{l_{r+1}} = 0$ since $P_{i_{r+1}} = 0$ and $\beta_l^\bullet$ is a map of complexes. Thus, $C(i_r) \cong C(l)$ and $\beta_l^\bullet$ is an isomorphism of complexes, contradicting the hypothesis of the lemma. Hence γ_r is an isomorphism and $P_{l_r} \cong P_{j_r}$.

Let $P_{l_{r+1}}$ be the component in degree $r+1$ of the complex $C(l)$, if $P_{l_{r+1}} \ncong P_{j_{r+1}}$ then the function

$$\varphi''_r : P_{j_r} \cong P_{l_r} \to P_{j_{r+1}}$$

in degree r of the complex $C(j_t)$ would factor as $\varphi_r'\gamma_{r+1}$, which implies that φ_r'' maps the top of P_{j_r} into the second composition factor isomorphic to S_{j_r} from the top of $P_{j_{r+1}}$, since by hypothesis $P_{j_{r+1}}$ is the predecessor of P_{j_r}, a contradiction. Therefore, $P_{l_{r+1}} \cong P_{j_{r+1}}$ and the map γ_{r+1} is an isomorphism. Arguing by induction, we find that $C(l) \cong C(j_t)$ and $\gamma_l^\bullet$ is an isomorphism of complexes, creating a contradiction to the assumption on $\gamma_l^\bullet$. Therefore,

$$\alpha^\bullet \in \mathrm{R}\ \mathrm{Hom}(C(i_r), C(j_t)) \setminus \mathrm{R}^2\ \mathrm{Hom}(C(i_r), C(j_t))$$

when the successor of i_r is j_t. This finishes the proof. □

Since we have a cyclic ordering on the edges, we induce a cyclic ordering on the vertices of Q_C in the obvious way.

Corollary 6.5. *Let i be a vertex in the quiver Q_C of* End(C) *and let k be its successor in the cyclic order defined above* (*see Lemma* 6.3). *Then there is one arrow from i to k.*

We relabel the edges of the Brauer tree $(T, \bar{m})$ with the elements of $\mathbb{Z}_n$, according with the cyclic order defined above (see Lemma 6.3). Remember that for each edge i we denote by e_i the multiplicity of the furthest end of i from the root v.

Lemma 6.6. *The quiver Q_C of* End(C) *consists of one arrow from the vertex i to its successor $i+1$ and a loop in each vertex j with $e_j > 1$.*

Proof. We only have to prove that the homomorphisms of Lemmas 6.1 and 6.4 generate End(C) as an algebra. Let $C(i)$ and $C(j)$ be direct summands of C. If $i = j$ then End($C(i)$) is spanned, as a K-algebra, by $\zeta^\bullet$ and $\eta^\bullet$ by Proposition 5.2. The function $\eta^\bullet$ is generated by the homomorphism of Lemma 6.1, so we only need to show that $\zeta^\bullet$ is generated. Denote by $\alpha_i^\bullet$ the arrow in Q_C from i to $i+1$, for all $i \in \mathbb{Z}_n$. We make an abuse of notation and denote maps and the corresponding arrows in the quiver with the same letters. Thus, $\alpha_i^\bullet$ represents the map described in Lemma 6.4. Then the function $\zeta_i^\bullet = \zeta^\bullet \in \mathrm{End}(C(i))$ defined in Proposition 5.2, satisfies

$$\zeta_i^\bullet = \lambda_i \alpha_i^\bullet \alpha_{i+1}^\bullet \cdots \alpha_{i-1}^\bullet$$

for some suitable scalar λ_i, for every i. Therefore, End($C(i)$) is generated by the homomorphisms of Lemmas 6.1 and 6.4 by Proposition 5.2.

Now assume that $i \neq j$, then $\dim_K \mathrm{Hom}(C(i), C(j)) = e_v$, where e_v is the multiplicity of the root vertex, by Lemma 5.1. We have a map from $C(i)$ to $C(j)$ given by $\alpha_i^\bullet \alpha_{i+1}^\bullet \cdots \alpha_{j-1}^\bullet$, the shortest path from i to j (in the cyclic ordering). Then

$$\mu_s^\bullet = \alpha_i^\bullet \alpha_{i+1}^\bullet \cdots \alpha_{j-1}^\bullet (\zeta_j^\bullet)^s$$

is a non-zero map in $\mathrm{Hom}(C(i), C(j))$ for all $s = 0, \ldots, e_v - 1$. Clearly, the functions $\mu_s^\bullet$, $s = 0, \ldots, e_v - 1$, are linear independent and therefore they generate $\mathrm{Hom}(C(i), C(j))$. This completes the proof. □

7. The quiver algebra of End(C)

In the previous section we discussed the shape of the quiver Q_C of the algebra $\mathrm{End}(C)$. In this section we will be interested on the relations on the above quiver, in order to give a description of the algebra $\mathrm{End}(C)$ by a quiver with relations. Let i be a vertex in Q_C, then there exist arrows $\alpha_{i-1}^\bullet : i-1 \to i$ and $\alpha_i^\bullet : i \to i+1$. If the multiplicity e_i of the corresponding edge i (the multiplicity of the furthest end of the edge i from the root) is greater than 1, we have a loop $\eta_i^\bullet$ at the vertex i by Lemma 6.6, where η_i represents the function $\eta \in \mathrm{End}(C(i))$ of Proposition 5.2.

Lemma 7.1. *The algebra* End(C) *is special biserial.*

Proof. From Definition 3.1, we have to check four conditions. For every vertex i in Q_C there are at most two arrows, $\alpha_i^\bullet$ and $\eta_i^\bullet$ when $e_i > 1$, starting at i, and there are at most two arrows, $\alpha_{i-1}^\bullet$ and $\eta_i^\bullet$ when $e_i > 1$, ending at i. Therefore, the conditions (1) and (1*) of Definition 3.1 are satisfied. To prove condition (2) and (2*) of Definition 3.1 it would be enough to show that $\alpha_{i-1}^\bullet \eta_i^\bullet = 0$ and $\eta_i^\bullet \alpha_i^\bullet = 0$ (only when $e_i > 1$). Let $C(i_r) = C(i)$ be the complex corresponding to i and $C(k) = C(i-1)$ be its predecessor, then the composition $\alpha_{i-1}^\bullet \eta_i^\bullet$ is given as follows:

$$
\begin{array}{ccccccccccccccc}
C(k): \cdots & \longrightarrow & 0 & \longrightarrow & P_{k_0} & \longrightarrow & P_{k_1} & \longrightarrow & P_{k_2} & \longrightarrow & \cdots & \longrightarrow & P_{k_r} & \longrightarrow & P_{k_{r+1}} \cdots \\
\downarrow{\scriptstyle \alpha_{i-1}^\bullet} & & \downarrow & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & & & \downarrow{\scriptstyle \alpha_{i-1}} & & \downarrow \\
C(i): \cdots & \longrightarrow & 0 & \longrightarrow & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow & \cdots & \longrightarrow & P_{i_r} & \longrightarrow & 0 \cdots \\
\downarrow{\scriptstyle \eta_i^\bullet} & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle \eta_i} & & \downarrow \\
C(i): \cdots & \longrightarrow & 0 & \longrightarrow & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow & \cdots & \longrightarrow & P_{i_r} & \longrightarrow & 0 \cdots
\end{array}
$$

where the functions $\eta_i^\bullet$ and $\alpha_{i-1}^\bullet$ are given as in Sections 5 and 6.1. If $\alpha_{i-1} = 0$ then we have nothing to prove. If $\alpha_{i-1} \neq 0$ and it is not the identity map, then k_r is the successor of i_r in the cyclic ordering of the vertex v_{r-1}, the furthest end of the edge i_{r-1}. Therefore, the composition $\alpha_{i-1}\eta_i = 0$, by the definition of these functions (see Sections 6.1 and 5). If α_{i-1} is the identity map then $C(k) = C(i_{r+1})$ and i_r is the predecessor of i_{r+1} in the cyclic ordering at v_r. From the construction of the complex $C(i_{r+1})$ in Section 4, the map $\psi_r : P_{i_r} \to P_{i_{r+1}}$ at level r of $C(i_{r+1})$ corresponds to a direct summand of the top of the radical of $P_{i_{r+1}}$. Thus, there exists a map $h_{r+1} : P_{i_{r+1}} \to P_{i_r}$ such that $\eta_i = \psi_r h_{r+1}$. This gives a contracting homotopy, and therefore $\alpha_{i-1}^\bullet \eta_i^\bullet$ is homotopic to zero.

Now we show that $\eta_i^\bullet \alpha_i^\bullet = 0$. Let $C(k) = C(i+1)$ be its successor, then the composition $\eta_i^\bullet \alpha_i^\bullet$ is given as follows:

$$\begin{array}{ccccccccccccccc}
C(i):\cdots & \longrightarrow & 0 & \longrightarrow & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow & \cdots & \longrightarrow & P_{i_r} & \longrightarrow & 0\ \cdots \\
\downarrow{\scriptstyle \eta_i^\bullet} & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle \eta_i} & & \downarrow \\
C(i):\cdots & \longrightarrow & 0 & \longrightarrow & P_{i_0} & \longrightarrow & P_{i_1} & \longrightarrow & P_{i_2} & \longrightarrow & \cdots & \longrightarrow & P_{i_r} & \longrightarrow & 0\ \cdots \\
\downarrow{\scriptstyle \alpha_i^\bullet} & & \downarrow & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & & & \downarrow{\scriptstyle \alpha_i} & & \downarrow \\
C(k):\cdots & \longrightarrow & 0 & \longrightarrow & P_{k_0} & \longrightarrow & P_{k_1} & \longrightarrow & P_{k_2} & \longrightarrow & \cdots & \longrightarrow & P_{k_r} & \longrightarrow & P_{k_{r+1}}\cdots
\end{array}$$

where the functions $\eta_i^\bullet$ and $\alpha_i^\bullet$ are given as in Sections 5 and 6.1. If $\alpha_i = 0$ then we have nothing to prove. If $\alpha_i \neq 0$ then α_i is not the identity map. Therefore, k_r is the predecessor of i_r in the cyclic ordering of the vertex v_{r-1}, the furthest end of the edge i_{r-1}. Therefore, the composition $\eta_i \alpha_i = 0$, by the definition of these functions (see Sections 6.1 and 5). Thus, the algebra End(C) is special biserial. □

Let i be a vertex in Q_C, using Lemma 7.1 and our previous work, it is easy to deduce the remaining relations on the quiver algebra of End(C). If the multiplicity of i is $e_i > 1$, then let $\zeta_i^\bullet = \zeta^\bullet$ as in Proposition 5.2. From the proof of Lemma 6.6, there exists a scalar multiple of $\zeta_i^\bullet$, that we denote by $\zeta_i^\bullet$ as well, such that

$$\zeta_i^\bullet = \alpha_i^\bullet \alpha_{i+1}^\bullet \cdots \alpha_{i-1}^\bullet .$$

From the discussion in Section 5 and in particular from the above Proposition 5.2, we know that there exists a scalar multiple of $\eta_i^\bullet$, that we denote by $\eta_i^\bullet$ as well, so that

$$(\eta_i^\bullet)^{e_i} = (\zeta_i^\bullet)^{e_v} = (\alpha_i^\bullet \alpha_{i+1}^\bullet \cdots \alpha_{i-1}^\bullet)^{e_v}, \tag{5}$$

where e_v is the multiplicity associated with the root v of the generalised Brauer tree T. If $e_i = 1$, then we obtain a monomial relation

$$(\alpha_i^\bullet \alpha_{i+1}^\bullet \cdots \alpha_{i-1}^\bullet)^{e_v} \alpha_i^\bullet = 0, \tag{6}$$

since Q_C does not have a loop at the vertex i by Lemma 6.6. Thus, by Lemma 7.1, these are the only possible relations, and our quiver algebra is completely defined. We summarise our analysis in the next theorem.

Theorem 7.2. *Let $\Lambda = (T, \bar{m})$ be a generalised Brauer tree algebra and let C its associated tilting complex of Lemma* 4.2. *Then the algebra* End(C) *is isomorphic to a quiver algebra KQ_C/I, where the quiver Q_C is as in Lemma* 6.6. *The ideal I is generated by the relations of Lemma* 7.1 *and the relations of Eqs.* (5) *and* (6).

One type of generalised Brauer tree that will be of particular interest to us is the "star" with n edges and $n+1$ vertices with vector of multiplicities $\bar{m}$. Following [9], we shall denote the corresponding generalised Brauer tree algebra as $B(n,\bar{m})$. The algebras of star type will turn to be very important in our description of the derived categories of generalised Brauer tree algebras.

Theorem 7.3. *Let Λ be a generalised Brauer tree algebra with generalised Brauer tree $(T,\bar{m})$. Let C its associated tilting complex as in Lemma 4.2. Then the algebra $\mathrm{End}(C)$ is a generalised Brauer tree algebra of star type, given by $B(n,\bar{m}')$.*

Proof. Let $B(n,\bar{m}')$ be a generalised Brauer tree algebra, with generalised Brauer tree S of star type with n edges. Let v be the centre of the star and denote by i the furthest end of the edge i, where the edges are cyclically ordered anticlockwise around v and the label i is an element of $\mathbb{Z}_n$. The vector $\bar{m}'$ is obtained from the original generalised Brauer tree algebra $(T,\bar{m})$, where the entries of $\bar{m}' = (e_0, e_1, \ldots, e_n)$ are ordered accordingly with the cyclic ordering of the edges of T as in Lemma 6.3, and $e_n = e_v$ is the multiplicity of the root. Then the algebra $B(n,\bar{m}')$ is completely defined. By Theorem 7.2, it is clear that $\mathrm{End}(C)$ is isomorphic to the algebra of star type $B(n,\bar{m}')$. □

Therefore, the algebras $(T,\bar{m})$ and $(B,\bar{m}')$ are derived equivalent by Theorem 7.3. We can now establish our generalisation of Theorem 4.2 of [9].

Theorem 7.4. *Every generalised Brauer tree algebra $(T,\bar{m})$ is derived equivalent to a generalised Brauer tree algebra of the form $B(n,\bar{m}')$, with the same number of edges and with the same multiplicities in some order determined by the inherent ordering on the edges of T.*

This theorem says that to study the derived equivalence classes of the generalised Brauer tree algebras, it is enough to study the derived equivalence classes of the generalised Brauer tree algebras of the form $B(n,\bar{m})$. But notice that we may have two generalised Brauer tree algebras of star type, with different vector of multiplicities and in the same derived equivalence class (see Section 9).

We can say more if we use the following corollary in Rickard's paper [9].

Corollary 7.5 (Rickard). *Let Λ and Γ be self-injective algebras. If Λ and Γ are derived equivalent then they are stably equivalent.*

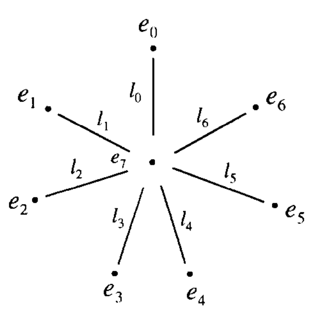
Since every generalised Brauer tree algebra is self-injective (they are even weakly symmetric), we can apply the previous corollary and get the next one, which generalises Theorem 1 in [6] for generalised Brauer tree algebras.

Corollary 7.6. *Every generalised Brauer tree algebra $(T,\bar{m})$ is stably equivalent to a generalised Brauer tree algebra of the form $B(n,\bar{m}')$, with the same number of edges* *and with the same multiplicities in some order determined by the inherent ordering on the edges of T.*

That is, the stable equivalence class of a generalised Brauer tree algebra can be represented by the number of edges of the Brauer tree and the vector of multiplicities $\bar{m}$, together with the inherent ordering on the edges of T. But notice that it may happen that two such representatives actually represent the same stable equivalence class (see Section 9).

7.1. An example

In this section we present a simple example of a generalised Brauer tree algebra in order to make the Theorem 7.4 clearer. We take advantage of the example in Section 6.1, since we already have worked out the ordering on the edges with respect to the root v. Let $\Lambda = (T, \bar{m})$ be a generalised Brauer tree algebra with generalised Brauer tree T as in the example of Section 6.1, the vector of multiplicities $\bar{m} = (e_0, e_1, \ldots, e_7)$ have its entries ordered accordingly, where e_7 is the multiplicity of the root v. We apply Theorem 7.4 (see Theorem 7.3 as well) to our algebra Λ and we find that Λ is derived equivalent to a generalised Brauer tree algebra $B(7, \bar{m}')$ with seven edges, which have its generalised Brauer tree S of star type, and the vector of multiplicities $\bar{m}' = (e_0, e_1, \ldots, e_7)$ have the same entries of $\bar{m}$ ordered accordingly with the cyclic ordering of the edges of T. The cyclic order on the edges of T gives the usual anticlockwise order on the edges of the star S around its centre. Then this generalised Brauer tree of star type is as follows:



Note that if we write the vector of multiplicities $\bar{m}'$ according to the cyclic order on the edges of S (instead of that of T), we obtain the vector

$$\bar{m}'' = (e_6, e_5, e_4, e_3, e_2, e_1, e_0, e_7),$$

where the last entry is the multiplicity of the root. Thus, we find that the vector of multiplicities of S is obtained from that of T by a permutation. This would give us an action of the symmetric group S_{n+1} on the derived equivalence classes of Brauer tree algebras with n edges. We study this action in the next section.

8. The cyclic ordering on $\bar{m}$

Let $\Lambda = (T, \bar{m})$ be a generalised Brauer tree algebra. From Theorem 7.4, Λ is derived equivalent to $B(n, \bar{m}')$, where n is the number of edges of T and $\bar{m}'$ is obtained from $\bar{m}$ by a permutation. By Lemma 6.3, there exists an inherent cyclic ordering on the edges of T (recall that the entries of the $(n+1)$-vector $\bar{m}$ are ordered accordingly, and the last entry is the multiplicity of the root of the tree T). Theorem 7.4 actually depends upon this ordering, in the following sense. We fix a tree S, which is a star with n edges. We order the edges in anticlockwise fashion around its centre (root). The algebra $B(n, \bar{m})$ has a generalised Brauer tree S and the multiplicities $e_0, \ldots, e_{n-1}$ correspond to the edges of S in this order, and e_n is the multiplicity of the centre (root) of the star.

In this section we discuss the importance of this ordering for the derived equivalence classes of generalised Brauer tree algebras. Namely, let $\bar{m}'$ be a $(n+1)$-vector obtained from $\bar{m}$ by permuting its entries. We want to know whether $B(n, \bar{m})$ and $B(n, \bar{m}')$ are derived equivalent. For instance, in the example of a generalised Brauer tree algebra Λ of Sections 6.1 and 7.1, we can choose as a root of the tree T the vertex with multiplicity e_3. Then the cyclic ordering on the corresponding vector of multiplicities of T is $e_0, e_4, e_1, e_2, e_5, e_6, e_7, e_3$ (where e_3 is the multiplicity of the root). So the algebra Λ of Section 7.1 is derived equivalent to $B(7, \bar{m}'')$, where $\bar{m}'' = (e_7, e_6, e_5, e_2, e_1, e_4, e_0, e_3)$, where the multiplicities are ordered according to the cyclic ordering on the star S with 7 edges and the multiplicity of the centre is e_3. Therefore, $B(7, \bar{m}'')$ and $B(7, \bar{m}')$ are derived equivalent, where $\bar{m}' = (e_6, e_5, e_4, e_3, e_2, e_1, e_0, e_7)$ (see example in Section 7.1).

Let S_{n+1} be the symmetric group on $n+1$ letters; then S_{n+1} acts on the derived equivalence classes of generalised Brauer tree algebras with n edges. The action is given by

$$\sigma B(n, \bar{m}) := B(n, \sigma \bar{m}),$$

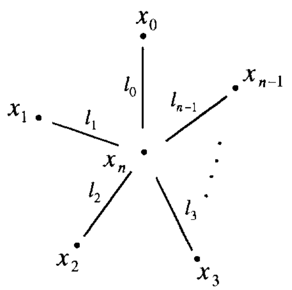
where $\sigma(e_0, e_1, \ldots, e_n) := (e_{\sigma 0}, e_{\sigma 1}, \ldots, e_{\sigma n})$. For example, since the edges of the star S are ordered cyclically around the centre, the algebra $B(n, \bar{m})$ is invariant under rotation of the star S. That is, let $\pi \in S_{n+1}$ so that

$$\pi \bar{m} = \pi(e_0, e_1, \ldots, e_{n-1}, e_n) = (e_1, e_2, \ldots, e_{n-1}, e_0, e_n),$$

then $B(n, \bar{m}) = \pi B(n, \bar{m})$. Therefore we have the following lemma.

Lemma 8.1. *The derived equivalence class of $B(n, \bar{m})$ is invariant under rotation of the star S.*

Given a generalised Brauer tree algebra $B(n, \bar{m})$, it corresponds to a star S with n edges. Let $l_0, l_1, \ldots, l_{n-1}$ denote the edges of S and let $x_0, x_1, \ldots, x_n$ be the corresponding vertices of S, where the centre is x_n and the vertices appear cyclically in anticlockwise fashion as follows:



Let e_i be the multiplicity associated with the vertex x_i and let

$$\bar{m} = (e_0, e_{n-1}, \ldots, e_1, e_n)$$

be the multiplicity vector of the algebra $B(n, \bar{m})$, in the cyclic ordering of Lemma 6.3. Recall from Section 4 that to construct our "canonical" tilting complex C for $B(n, \bar{m})$ (see Lemma 4.2) we choose an arbitrary but fixed vertex v of S and we say that this is the root of S. We apply Theorem 7.4 to $B(n, \bar{m})$, and we obtain that End(C) is again an algebra of star type $B(n, \bar{m}')$, where $\bar{m}' = \theta\bar{m}$, for some $\theta \in S_{n+1}$. For instance, if $v = x_n$, the centre of the star, then the permutation θ is a reflection of the star S. In fact, let $B(n, \bar{m}')$ be the generalised Brauer tree algebra of star type resulting by applying Theorem 7.4 with root x_n. Then the anticlockwise ordering of the multiplicities associated with the edges around the centre of $B(n, \bar{m}')$ is $e_0, e_{n-1}, \ldots, e_1$, since $\bar{m} = (e_0, e_{n-1}, \ldots, e_1, e_n)$. Then the vector of multiplicities $\bar{m}'$ with its entries ordered accordingly with the cyclic order of Lemma 6.3 is $(e_0, e_1, \ldots, e_{n-1}, e_n)$ (see Lemma 8.1), where e_n is the multiplicity of the root. Therefore, the action of S_{n+1} in this case is given by a reflection ρ, acting as above. Then we have proven the following lemma.

Lemma 8.2. *The derived equivalence class of $B(n, \bar{m})$ is invariant under rotation and reflection of the star S.*

Now suppose that we have chosen the root $v = x_i$ different from the centre x_n. Then by Theorem 7.4, we have that End(C) $\cong B(n, \bar{m}')$, where the centre of the star has multiplicity e_i and the multiplicities of the outer vertices are ordered according to the cyclic order of the edges relative to the root x_i (see Section 6.1). This order gives $l_{i-1}, l_{i-2}, \ldots, l_0, l_{n-1}, \ldots, l_{i+1}, l_i$, and the corresponding multiplicities are $e_{i-1}, e_{i-2}, \ldots, e_0, e_{n-1}, \ldots, e_{i+1}, e_n$. Then the multiplicities of the star of $B(n, \bar{m}')$ around the centre are ordered anticlockwise as follows $e_0, e_{n-1}, \ldots, e_{i+1}, e_n, e_{i-1}, \ldots, e_1$, and the multiplicity of the centre is e_i. If we write the entries of $\bar{m}'$ according to the cyclic order of Lemma 6.3 and using Lemma 8.1, we obtain

$$\bar{m}' = (e_0, e_1, \ldots, e_{i-1}, e_n, e_{i+1}, \ldots, e_{n-1}, e_i).$$

This corresponds to a transposition $(i\ n)$ followed by a reflection ρ of the star S, module some rotation π as in Lemma 8.2, which does not modify the algebra $\rho(i\ n)B(n,\bar{m})$. This implies that the derived equivalence class of $B(n,\bar{m})$ is invariant under the action of permutations of the form $\rho(i\ n) \in S_{n+1}$. But this is enough to prove the following theorem.

Theorem 8.3. *The derived equivalence class of $B(n,\bar{m})$ is invariant under the action of S_{n+1}.*

Proof. By the discussion above, we have that the derived equivalence class of $B(n,\bar{m})$ is invariant under the action of permutations of the form $\rho(i\,n)$ for all $i = 0,\dots,n-1$, reflection ρ and rotations π as in Lemma 8.2. By the same Lemma 8.2, the derived equivalence class of $\rho(i\,n)B(n,\bar{m})$ is invariant under the action of a reflection ρ. Therefore, the derived equivalence class of $B(n,\bar{m})$ is invariant under the action of transpositions of the form $(i\ n)$ for all $i = 0,\dots,n-1$, since ρ^2 is the identity. Every transposition $(i\ j) \in S_{n+1}$ is a product of transposition of the above type. Namely $(i\,j) = (i\,n)(j\,n)(i\,n)$. Therefore, the derived equivalence class of $B(n,\bar{m})$ is invariant under the action of S_{n+1}, since every permutation is a product of transpositions. □

This theorem allow us to improve substantially Theorem 7.4, since the order defined in Section 6.1 on the multiplicities is irrelevant to the derived equivalence classes of the Brauer tree algebras.

Theorem 8.4. *Every generalised Brauer tree algebra $(T,\bar{m})$ is derived equivalent to a generalised Brauer tree algebra of the form $B(n,\bar{m}')$, with the same number of edges and with the same multiplicities, regardless of the inherent ordering on the edges.*

We can say more if we use Rickard's Corollary 7.5 (see [9]). This gives a better generalisation of Theorem 1 in [6] for generalised Brauer tree algebras.

Corollary 8.5. *Every Brauer tree algebra $(T,\bar{m})$ is stably equivalent to a Brauer tree algebra of the form $B(n,\bar{m}')$, with the same number of edges and with the same multiplicities, regardless of the ordering on the entries of $\bar{m}$.*

9. A complete characterisation

Let $\Lambda = (T,\bar{m})$ be a generalised Brauer tree algebra. We have seen in Theorem 7.4 that Λ is derived equivalent to a generalised Brauer tree algebra $B(n,\bar{m}')$ of star type, where n is the number of indecomposable projective modules of Λ (the number of edges of T, recall that n is invariant under derived equivalence). Hence, we can represent each derived equivalence class of generalised Brauer tree algebras with an algebra of star type. Moreover, in Theorem 8.4 we have shown that the order on the entries of the vector of multiplicities has no significance for the derived equivalence classes. That is, the generalised Brauer tree algebras $B(n,\bar{m})$ and $B(n,\sigma\bar{m})$ are derived equivalent for any permutation $\sigma \in S_{n+1}$ (see Theorems 8.3 and 8.4). We denote by $B(n,\hat{m})$ the derived equivalence class of $B(n,\bar{m})$, where $\hat{m} := \{e_0, e_1, \ldots, e_n\}$ is the vector of multiplicities $\bar{m} = (e_0, e_1, \ldots, e_n)$ forgetting the order on its entries. We call $\hat{m}$ the set of multiplicities and we have

$$B(n,\sigma\bar{m}) \in B(n,\hat{m}) \quad \text{for all } \sigma \in S_{n+1}.$$

Given two different sets of multiplicities $\hat{m} = \{e_0, e_1, \ldots, e_n\}$ and $\hat{m}' = \{e_0', e_1', \ldots, e_n'\}$, we want to answer the following very specific question: is $B(n,\hat{m}) = B(n,\hat{m}')$? In order to answer this question we have to look for invariants under derived equivalence. We will be particularly interested in the Hochschild cohomology ring associated with a finite-dimensional algebra. We recall that two derived equivalent finite-dimensional algebras Λ and Γ have isomorphic Hochschild cohomology ring. Namely, $\mathrm{H}(\Lambda) \cong \mathrm{H}(\Gamma)$ as $\mathbb{Z}$-graded algebras. In fact, it would be enough to consider the following particular case.

Proposition 9.1 (Rickard [10]). *If Λ and Γ are derived equivalent algebras, then the centres $Z(\Lambda)$ and $Z(\Gamma)$ are isomorphic (as algebras).*

For the rest of this section we shall study the centre of a given generalised Brauer tree algebra $\Lambda = (T,\bar{m})$. By the above Proposition 9.1 and Theorem 8.4. The algebra $Z(\Lambda)$ is isomorphic to the algebra $Z(B(n,\bar{m}))$, for a generalised Brauer tree algebra of star type and n is the number of edges of T.

We firstly consider the case $n = 1$. That is, our algebra $B = B(1,\bar{m})$ has only one projective indecomposable module and its generalised Brauer tree has only one edge (see Section 2). We draw this generalised Brauer tree below:

$$e_1 \bullet \xrightarrow{\quad l_0 \quad} \bullet\, e_0$$

where the multiplicity associated with the vertex v_i is e_i. We select the vertex v_1 to be the root of this tree (see Section 4), that is, v_1 is the centre of this "degenerated" star. Then the algebra $B(1,(e_0,e_1))$ is local and by construction it is commutative (abelian). Therefore, the centre of the algebra coincides with the algebra itself:

$$Z(B(1,(e_0,e_1))) = B(1,(e_0,e_1)).$$

So in this case, the derived equivalence classes reduces to isomorphism (Morita equivalence) classes of generalised Brauer tree algebras $B(1,\bar{m})$ by Proposition 9.1. Therefore, we see that if $B(1,\bar{m})$ is derived equivalent to $B(1,\bar{m}')$ then the sets $\hat{m}$ and $\hat{m}'$ have to be equal.

Lemma 9.2. *Let $B(1,\hat{m})$ and $B(1,\hat{m}')$ be two derived equivalence classes of generalised Brauer tree algebras. Then $B(1,\hat{m}) = B(1,\hat{m}')$ if and only if $\hat{m} = \hat{m}'$.*

Our analysis allow us to calculate the dimension of the algebra $Z(B(1,\bar{m})) = B(1,\bar{m})$ over K.

Lemma 9.3. *The generalised Brauer tree algebras $B(1,\bar{m})$ satisfy*

$$\dim_K Z(B(1,\bar{m})) = \dim_K B(1,\bar{m}) = e_0 + e_1,$$

unless $B(1,\bar{m})$ is the "trivial" Brauer tree algebra $B(1,(1,1))$. In this case

$$\dim_K B(1,(1,1)) = 1.$$

We continue with the general case. Let $\Lambda = (T,\bar{m})$ be a generalised Brauer tree algebra with its generalised Brauer tree T having at least two edges. That is, Λ has at least two non-isomorphic indecomposable projective modules. Then Λ is derived equivalent to $B = B(n,\bar{m})$ with $n > 1$, and $Z(\Lambda) \cong Z(B)$. Note that in this case the centre $Z(B)$ of B is properly contained in B. We proceed to identify the elements of $Z(B)$.

Let $x_0, x_1, \dots, x_{n-1}$ be a complete list of primitive orthogonal idempotents of B. These idempotents are in correspondence with the vertices $x_0, x_1, \dots, x_{n-1}$ in the Gabriel quiver Q_B (see [5]) of B (we identify the vertices of Q_B with elements of B), which at the same time are in correspondence with the edges $l_0, l_1, \dots, l_{n-1}$ of the star (tree) of B (see Section 6). The outer multiplicities $e_0, e_1, \dots, e_{n-1}$ of the star of B are also in correspondence with the edges $l_0, l_1, \dots, l_{n-1}$.

Under this setting we know the exact shape of the quiver Q_B. By Lemma 6.6 (see also Sections 2 and 3) we know that the quiver Q_B consist of n vertices $x_0, x_1 \dots, x_{n-1}$ (numbered accordingly with the cyclic order of Lemma 6.3 of the corresponding edges around the centre of the star of B). We relabel the multiplicities accordingly (that is, clockwise around the centre). There is an arrow $\alpha_i : i \to i+1$ for all $i \in \mathbb{Z}_n$ and there is a loop η_j (see Corollary 6.2) at x_j if and only if its corresponding multiplicity $e_j > 1$.

We also know the relations in the quiver Q_B since B is a generalised Brauer tree algebra (see also Theorem 7.2). Let $\zeta_i = \alpha_i\alpha_{i+1}\cdots\alpha_{n-1}\alpha_0\cdots\alpha_{i-1}$; then for every multiplicity $e_j > 1$ we have $\eta_j^{e_j} = \zeta_j^{e_n}$, where e_n is the multiplicity of the centre of the star of B. The remaining relations are $\zeta_i^{e_n}\alpha_i = 0$ if $e_i = 1$; $\alpha_{i-1}\eta_i = \eta_i\alpha_i = 0$ if $e_i > 1$. Then B is special biserial (see Section 3 and Lemma 7.1).

Let x_j be a vertex of Q_B with multiplicity $e_j > 1$. Then there is a loop η_j at x_j and $\eta_j \in Z(B)$. In fact, by the relations in the quiver of B described above we know that $\alpha_{j-1}\eta_j = \eta_j\alpha_j = 0$, therefore $\eta_j\gamma = \gamma\eta_j$ for all $\gamma \in B$.

We can identify more elements of the centre of B if we consider the (right) socle of B. The (right) socle of B is spanned, as a vector space, by $\zeta_i^{e_n}$ for all $i \in \mathbb{Z}_n$. That is, Soc B is spanned by the longest paths in Q_B starting at each vertex x_i (see [5]). Note that in case $e_i > 1$ then $\eta_i^{e_i} = \zeta_i^{e_n}$, then the right and left socles of B coincide (the left socle of B is spanned by the longest paths ending at each vertex x_i for all $i \in \mathbb{Z}_n$) and it is a bilateral ideal of B. We call this ideal simply the socle of B. Let $\zeta_i^{e_n}$ be one of the spanning elements of the socle of B and x_i the corresponding vertex; then

$$x_i\zeta_i^{e_n} = \zeta_i^{e_n}x_i = \zeta_i^{e_n},$$

and therefore $\gamma\zeta_i^{e_n} = \zeta_i^{e_n}\gamma$, for all $\gamma \in B$. In particular, we find that

$$\operatorname{Soc} B \leq Z(B).$$

Now consider the element $\xi = \sum_{i\in\mathbb{Z}_n} \zeta_i$. Then $\xi \in Z(B)$, since clearly ξ commutes with every vertex x_i and with every arrow α_i or loop η_i of Q_B.

Lemma 9.4. *Let $B = B(n, \bar{m})$ be a generalised Brauer tree algebra of star type with $n \geq 2$. Then the centre $Z(B)$ of B is generated by* Soc B, *ξ and loops η_j for each $e_j > 1$.*

Proof. Let $\varepsilon \in Z(B)$ and define $\varepsilon_i = x_i\varepsilon$ for all $i \in \mathbb{Z}_n$. Then

$$\varepsilon = 1_B\varepsilon = \left(\sum_{i\in\mathbb{Z}_n} x_i\right)\varepsilon = \sum_{i\in\mathbb{Z}_n} x_i\varepsilon = \sum_{i\in\mathbb{Z}_n} \varepsilon_i,$$

where 1_B is the identity element of B. Since $\varepsilon_i \in x_iB$ and $\varepsilon_i = \varepsilon x_i$ ($\varepsilon \in Z(B)$) then $\varepsilon_i \in x_iBx_i$ for all $i \in \mathbb{Z}_n$. Note that the vector space x_iBx_i is spanned by the vertex x_i, the loop η_i (if $e_i > 1$) and its powers together with the path ζ_i and its powers. Thus,

$$\varepsilon_i = \lambda_0^i x_i + \sum_{s=1}^{e_i} \lambda_s^i \eta_i^s + \sum_{t=1}^{e_n} \lambda_t^i \zeta_i^t.$$

Since $\eta_i \in Z(B)$ and $\zeta_i^{e_n} \in Z(B)$, we subtract from ε_i all the summands corresponding to powers of η_i and the summand corresponding to $\zeta_i^{e_n}$ and we get

$$\varepsilon_i' = \sum_{s=0}^{e_n-1} \lambda_s^i \zeta_i^s,$$

where $\zeta_i^0 = x_i$. We do this for all $i \in \mathbb{Z}_n$. Hence

$$\varepsilon' = \sum_{i\in\mathbb{Z}_n} \varepsilon_i' \in Z(B).$$

Let $\alpha_i : i \to i+1$ be an arrow in Q_B so

$$\begin{aligned} \varepsilon'\alpha_i = \varepsilon_i'\alpha_i &= \left(\sum_{s=0}^{e_n-1} \lambda_s^i \zeta_i^s\right)\alpha_i \\ &= \sum_{s=0}^{e_n-1} \lambda_s^i (\zeta_i^s \alpha_i) \\ &= \sum_{s=0}^{e_n-1} \lambda_s^i (\alpha_i \zeta_{i+1}^s). \end{aligned}$$

Now we use $\varepsilon' \in Z(B)$.

$$\begin{aligned}\varepsilon'\alpha_i = \alpha_i\varepsilon' &= \alpha_i\varepsilon'_{i+1}\\ &= \alpha_i\left(\sum_{s=0}^{e_n-1}\lambda_s^{i+1}\zeta_{i+1}^s\right)\\ &= \sum_{s=0}^{e_n-1}\lambda_s^{i+1}(\alpha_i\zeta_{i+1}^s).\end{aligned}$$

Since $\{\alpha_i\zeta_{i+1}^s : 0 \le s \le e_n - 1\}$ is a linear independent set, we find that $\lambda_s^i = \lambda_s^{i+1}$ for all $s = 0, \ldots, e_n - 1$. Therefore, by induction, we see that

$$\varepsilon'_j = \sum_{s=0}^{e_n-1}\lambda_s\zeta_j^s \qquad \text{for all } j \in \mathbb{Z}_n,$$

where the coefficients λ_s are the same for all $j \in \mathbb{Z}_n$. Therefore,

$$\varepsilon' = \sum_{s=0}^{e_n-1}\lambda_s\xi^s,$$

where $\xi^0 = 1_B$. That is, our original ε is in the span of Soc B, ξ and loops η_j for each $e_j > 1$. □

Before we proceed to compare the centre of $\Lambda = (T, \bar{m})$ with another generalised Brauer tree algebra with different set of multiplicities, we calculate the dimension of $Z(\Lambda)$.

Theorem 9.5. *Let $\Lambda = (T, \bar{m})$ be a generalised Brauer tree algebra with vector of multiplicities $\bar{m} = (e_0, e_1, \ldots, e_n)$. Then*

$$\dim_K Z(\Lambda) = \dim_K B(n, \bar{m}) = \sum_{i=0}^{n} e_i,$$

unless Λ is the "trivial" Brauer tree algebra $B(1,(1,1))$. In this case

$$\dim_K B(1,(1,1)) = 1.$$

Proof. If Λ is the "trivial" Brauer tree algebra or $n = 1$, then the Theorem is true by Lemma 9.3. Assume $n > 1$ and consider the algebra $B = B(n, \bar{m})$ which have the same centre as Λ by Proposition 9.1. Let x_i be a vertex in the quiver of Q_B (a primitive idempotent of B). If its corresponding multiplicity $e_i > 1$, then in the vector space x_iBx_i we have the linear independent set $\{\eta_i^s : 1 \le s \le e_i\}$ contained in $Z(B)$, where η_i is the loop at x_i. If $e_i = 1$ then we have the corresponding element $\zeta_i^{e_n}$ in the socle of B (the longest path starting at x_i, see notation in Lemma 9.4). Therefore, in each x_iBx_i we have e_i linear independent elements in $Z(B)$, for $i = 0, \ldots, n-1$, and these sets are linear independent between them. Consider the element $\xi = \sum_{i\in\mathbb{Z}_n} \zeta_i$ of Lemma 9.4; then $\{\xi^i : 0 \le i \le e_n - 1\}$ is a linear independent set, where $\xi^0 = 1_B$. This set contains e_n elements and it is linear independent of the previous sets (note that $\xi^{e_n} \in \text{Soc } B$) by Lemma 9.4. This elements span $Z(B)$ by Lemma 9.4. Thus,

$$\dim_K Z(\Lambda) = \dim_K Z(B(n,\bar{m})) = \sum_{i=0}^{n} e_i. \qquad \square$$

Let $B = B(n,\bar{m})$ and $B' = B(n,\bar{m}')$ be representatives of the derived equivalence classes of generalised Brauer tree algebras $B(n,\hat{m})$ and $B(n,\hat{m}')$ respectively, with $n > 1$. Assume that $B(n,\hat{m}) = B(n,\hat{m}')$, our goal is to prove that $\hat{m} = \hat{m}'$. Firstly, note that there exists an isomorphism of K-algebras

$$\varphi : Z(B) \to Z(B')$$

between the algebras $Z(B)$ and $Z(B')$ by Proposition 9.1. Recall that for a generalised Brauer tree algebra of star type we have that Soc B is a bilateral semi-simple ideal and Soc $B \le Z(B)$ by Lemma 9.4. Moreover, the left and right socles of $Z(B)$ coincide ($Z(B)$ is commutative) and Soc $Z(B)$ = Soc B as bilateral ideals of B. Therefore, $\varphi(\text{Soc } B) = \text{Soc } B'$ and φ induces an isomorphism

$$\bar{\varphi} : Z(B)/\text{Soc B} \to Z(B')/\text{Soc } B'.$$

We may give a description of $\overline{Z(B)} := Z(B)/\text{Soc } B$ as a quiver with relations. The quiver $Q_{\overline{Z(B)}}$ of $\overline{Z(B)}$ consists of one vertex x_1 ($Z(B)$ is local) and for each multiplicity e_i in $\bar{m}$ with $e_i > 1$ a loop l_i. The ideal of relations I is generated by $l_i^{e_i} = 0$ and $l_i l_j = 0$ for $i \ne j$. The map

$$\psi : \overline{Z(B)} \to KQ_{\overline{Z(B)}}/I,$$

given by $1_B \mapsto x_1$; $\eta_i \mapsto l_i$ is an isomorphism of K-algebras, where 1_B is the identity element of B and η_i is the loop in Q_B corresponding to the multiplicity $e_i > 1$ in $\bar{m}$. Note that if $e_n > 1$, then $\xi \mapsto l_n$ (see Lemma 9.4).

The radical Rad $\overline{Z(B)}$ of $\overline{Z(B)}$ is a bilateral ideal of $\overline{Z(B)}$ and since $\bar{\varphi}$ is an isomorphism we have $\bar{\varphi}(\text{Rad } \overline{Z(B)}) = \text{Rad } \overline{Z(B')}$ (see for example [2]). Note that Rad $\overline{Z(B)}$ is a direct sum of uniserial modules, one for each loop l_i (one for each multiplicity $e_i > 1$ in $\bar{m}$). The number of this summands is an invariant under isomorphism. Therefore, the number of multiplicities e_i in $\hat{m}$ greater than 1, is the same as the number of multiplicities e'_j in $\hat{m}'$ greater than 1. Moreover, the length of each uniserial summand of Rad $\overline{Z(B)}$ is also an invariant under isomorphism. The length of the uniserial corresponding to the loop l_i is $e_i - 1$. Then, there exists a multiplicity e'_j (greater than 1) in $\bar{m}'$ such that $e_i = e'_j$. Thus, there exists a permutation $\sigma \in S_{n+1}$ so that

$$e_i = e'_{\sigma i} \quad \text{for all } i = 0, 1, \ldots, n.$$

Our considerations together with Lemma 9.2 prove the main theorem of this article, this gives a complete characterisation of the derived equivalence classes of generalised Brauer tree algebras.

Theorem 9.6. *Let $B(n,\hat{m})$ and $B(n',\hat{m}')$ be two derived equivalence classes of generalised Brauer tree algebras. Then $B(n,\hat{m}) = B(n',\hat{m}')$ if and only if $n = n'$ and $\hat{m} = \hat{m}'$.*

We can now improve Theorems 7.4 and 8.4 to a complete characterization of the derived equivalence classes of the generalised Brauer tree algebras. This is also a complete generalisation of Theorem 4.2 of [9].

Theorem 9.7. *Up to derived equivalence, a generalised Brauer tree algebra is determined by the number of edges of its generalised Brauer tree and the set of multiplicities.*

Therefore, to study the derived category of a generalised Brauer tree algebra $(T,\bar{m})$, it is enough to study the derive category of the class $B(n,\hat{m})$, where two different sets $\hat{m} \neq \hat{m}'$ give not triangle-equivalent categories. Using Corollary 7.5 we get the following corollary which improves Corollarys 7.6 and 8.5. It also generalises Theorem 1 in [6] for generalised Brauer tree algebras.

Corollary 9.8. *The stable equivalent classes of generalised Brauer tree algebras split into the derived equivalent classes $B(n,\hat{m})$.*

That is, we know that every stable equivalence class of generalised Brauer tree algebras is built up of "bricks" of the form $B(n,\hat{m})$. We do not know what "bricks" are in the same stable equivalence class. To consider this problem it would be necessary to characterize the Auslander–Reiten quiver of a generalised Brauer tree algebra. This has been done in the seminal work of Gabriel and Riedtmann [6] for Brauer tree algebras. They characterised the stable equivalence classes of Brauer tree algebras (the generalised Brauer tree algebras of finite type). Using these results, Rickard [9] showed that the Brauer tree algebras which are stably equivalent are in fact derived equivalent. That is, for Brauer tree algebras, the stable equivalence classes are built up of only one "brick". The general case is unknown. However, our feeling is that this is the case in general; the following conjecture is felt outside of the scope of this work. It would generalise Theorem 2 of [6] to the generalised Brauer tree algebras case.

Conjecture 9.9. *The derived equivalence classes $B(n,\hat{m})$ coincide with the stable equivalence classes of generalised Brauer tree algebras.*

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